

The Shapley Value

Game Theory Course:
Jackson, Leyton-Brown & Shoham

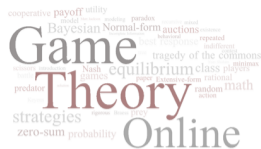
Coalitional or Cooperative Games



- Question: what is a 'fair' way for a coalition to divide its payoff?
- This depends on how we define 'fairness.'
- One Approach: identify **axioms** that express properties of a fair payoff division.

The Shapley Value

- Lloyd Shapley's idea: members should receive payments or shares proportional to their marginal contributions



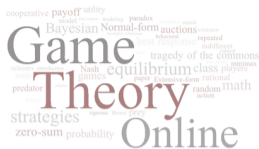
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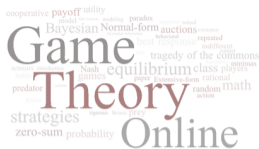
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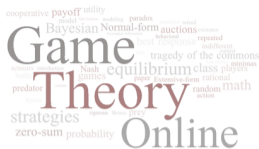
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 - Cannot pay everyone their marginal Contribution!
- We will have to use some weighting system - how should it be designed?
- Shapley's axioms give us one answer...



Symmetry



- i and j are **interchangeable** relative to v if they always contribute the same amount to every coalition of the other agents.
 - for all S that contains neither i nor j , $v(S \cup \{i\}) = v(S \cup \{j\})$.

Axiom (Symmetry)

For any v , if i and j are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$.

- Interchangeable agents should receive the same shares/payments.

Dummy Players

- i is a **dummy player** if the amount that i contributes to any coalition is 0.
 - for all S : $v(S \cup \{i\}) = v(S)$.

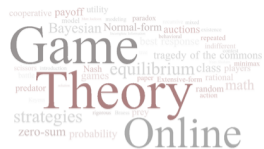
Axiom (Dummy player)

For any v , if i is a dummy player then $\psi_i(N, v) = 0$.

- Dummy players should receive nothing.



Additivity



- If we can separate a game into two parts $v = v_1 + v_2$, then we should be able to decompose the payments:

Axiom (Additivity)

For any two v_1 and v_2 , $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$ for each i , where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S .

Shapley Value

Given a coalitional game (N, v) , the **Shapley Value** divides payoffs among players according to:

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \left[v(S \cup \{i\}) - v(S) \right].$$

for each player i .



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Theorem

Given a coalitional game (N, v) , there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms: the Shapley Value

Understanding the Shapley Value



$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \left[v(S \cup \{i\}) - v(S) \right].$$

This captures the “marginal contributions” of agent i , averaging over all the different sequences according to which the grand coalition could be built up.

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- For any such sequence, look at agent i 's marginal contribution when added: $[v(S \cup \{i\}) - v(S)]$.

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- Weight this quantity by the $|S|!$ ways the set S could have been formed prior i 's addition

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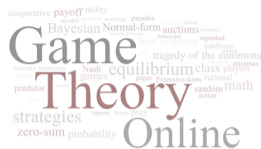


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- Weight this quantity by the $|S|!$ ways the set S could have been formed prior i 's addition and by the $(|N| - |S| - 1)!$ ways the remaining players could be added.

Understanding the Shapley Value

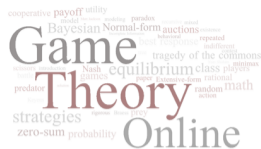


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- Sum over all possible sets S and average by dividing by $|N|!$: the number of possible orderings of all the agents.

Understanding the Shapley Value:



